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A MATHEMATICAL MODEL OF NOZZLE BLOCKAGE BY FREEZING—II. TURBULENT FLOW

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Abstract—Hot molten metal flowing through a cylindrical pipe is frozen by the cold walls of the pipe and forms a solid crust on the walls. A model is presented to describe the growth of this crust given that the flow through the pipe is turbulent. Criteria which determine whether or not the pipe will block are investigated and some results from the theoretical analysis are presented for flows with Reynolds number between 2000 and 100,000 appropriate to those fluids with Prandtl numbers in the range 0.007–0.1.

NOMENCLATURE

$A_n(Pr, Re)$,	nth eigenconstant in series (43);
B,	dimensionless wall temperature;
$c(Pr, Re)$,	turbulent flow parameter;
C,	specific heat;
$C_n(Pr, Re)$,	nth eigenconstant in series (37);
d, d_1,	perturbations of D_s;
$D(t)$,	dimensionless crust growth parameter;
$D_s, D_{su}, D_{ss}, D_{sc}, D_{sz}$,	steady state variants of D;
$f(R)$,	axial velocity function;
f',	Blasius friction factor;
$\hat{f}(z,t)$,	heat flux used in stability analysis;
$\hat{f}_s, \hat{f}_{su}, \hat{f}_{ss}$,	steady state variants of $\hat{f}$;
$g(R)$,	turbulent thermal diffusivity;
$\hat{g}(z)$,	function used in stability analysis;
$G(Pr, Re)$,	turbulent flow parameter;
Gz,	$= Pr_0^4/4\rho v\kappa\, l_0^2$, Graetz Number;
$H(Re)$,	turbulent flow parameter;
k_f, k_s,	thermal conductivities of fluid and solid;
l_0,	length of cylinder;
$\ln(x)$,	natural logarithm function;
L,	latent heat of fusion of fluid;
$p(z,t)$,	fluid pressure;
P,	pressure drop over pipe;
Pe, Pe_0,	Peclet number and initial Peclet number;
Pr,	$= v/\kappa$, Prandtl number;
$2\pi Q(t)$,	inlet volume flow rate;
r,	radial coordinate;
r_0,	radius of cylinder;
R,	'crust free' radial coordinate;
$R_n(R)$,	nth eigenfunction in series (37);
Re, Re_0,	Reynolds number and initial Reynolds number;
St,	$= C(T_0 - T_m)/L$, Stefan number;
t,	time;
$T(r,z,t)$,	temperature;
T_m, T_0, T_w,	melting, entry and wall temperatures;
$u(r,z,t)$,	radial velocity component;
$w(r,z,t)$,	axial velocity component;
$\bar{w}(z,t)$,	bulk axial velocity component;
z,	axial coordinate.

Greek symbols

α,	dimensionless pipe length;
$\Gamma(x)$,	gamma function of x,
$\delta(z,t)$,	crust position;
$\delta_s(z)$,	steady state crust position;
$\theta(r,z,t)$,	dimensionless temperature;
κ,	$= k_f/\rho C$, thermal diffusivity of fluid;
λ_n,	nth eigenvalue in series (37) and (43);
$\mu(D_s)$,	function used in stability analysis;
v,	kinematic viscosity of fluid;
ρ,	density of both fluid and solid;
σ,	decay rate used in stability analysis.

Superscript

$*$,	denotes a dimensional variable.

INTRODUCTION

THE SOLIDIFICATION of a hot liquid as it flows through a cold pipe has been of interest for some years now and many models have been proposed. The problem is of practical significance to many engineers and of particular interest to those in metals technology. In this field it is common for liquid metals to be passed through pipes either to be directed or to be used in some continuous casting process. The flow of the liquid metal is likely to be turbulent and consequently it is difficult to analyse the flow and heat transfer. Turbulence affects these variables through viscosity and conductivity and so it is apparent from the outset that solutions will depend upon the Reynolds number of the flow and the Prandtl number of the fluid. Because of this, it is necessary to restrict numerical analysis of the problem to certain ranges of Reynolds and Prandtl numbers although the mathematical analysis should remain valid for any turbulent flows, within the confines of the model.

There appear to be two main approaches to this type of problem. A general approach, adopted by Szekely and DiNovo [1], uses many mathematical equations familiar in heat transfer theory to produce numerical results directly. While this approach takes into account the fluid region before solidification and finite thickness of the pipe wall, little analysis is performed. A steady state model by Thomason *et al.* [2] uses heat transfer coefficients, in common with [1], to develop a theory in conjunction with some experimental work on water. The mathematical model is simpler and has more in common with our model than that of Szekely. These models are based largely upon Nusselt number formulations.

The second approach, based rather upon the Graetz number, is the one that we shall adopt. The Graetz Problem involves the solution of the steady state radial conduction and axial convection heat equation in terms of an infinite series of eigenfunctions with associated eigenvalues and eigenconstants. The analysis for laminar flow has been known for some time now [3] and more recently, Sleicher *et al.* [4] and Notter and Sleicher [5] presented an analysis for the turbulent Graetz problem in a constant diameter cylinder wherein may be found tables of the relevant eigenquantities and formulae for asymptotic forms.

A steady state model proposed by Shibani and Özisik [6] drew upon the work by Sleicher, recalculating some of the coefficients, and applied it to the freezing of fluids with Prandtl numbers between 0 and 1000 although the applicability of their analysis to a fluid with a Prandtl number of zero is suspect since the assumption of a large Peclet number is no longer valid. A later paper by Cho and Özisik [7] dealt with transient freezing, i.e. the development of the solidified crust in time, by integral transform methods. It is difficult, however, to ascertain blockage criteria with these models since the results seem to indicate an exponential type approach to pipe closure, thus necessitating the use of some *ad hoc* definition of blockage, whereas the present approach allows for the possibility of complete blockage. It should be pointed out that the fine details of blockage are complex and other models should be applied for this sort of information, the main interest lying not in the mechanism, but rather the fact, of blockage.

THE MODEL

The analysis in this work relates to the simplified model shown in Fig. 1. A hot fluid of constant density, viscosity, thermal conductivity and specific heat is forced through a cold pipe of finite length and radius by a pressure drop maintained over the length of the pipe. The pipe wall is held at a constant temperature, below the melting point of the fluid, providing an isothermal boundary for the fluid whilst it is inside the pipe. The fluid temperature on entry to the pipe is constant and is greater than the melting temperature of the fluid. The flow is considered to be turbulent, fully developed and radially symmetric, allowing us to dispense with angular components and dependencies in the cylindrical coordinate system used in the analysis. This last assumption is valid if buoyancy effects and free convection are neglected. Since the flow is turbulent, conduction of both momentum and heat in the axial direction will be ignored as being negligible in comparison with radial conduction, allowing us to drop the axial double derivatives, an assumption valid for a large Peclet number.

Initially there is no solidified layer but as time passes, the fluid loses heat to the pipe wall and begins to freeze inwardly. This solid layer will, like the fluid, have constant specific heat, thermal conductivity and a density equal to that of the fluid. If the Stefan number of the system is sufficiently small, then the flow of the fluid can be described as quasi-steady allowing us to neglect the time rate of change of temperature and momentum in the fluid equations. This effectively means that the fluid flow and temperature fields will change more quickly than the position of the solid/liquid interface.

The mathematical model is developed in the following manner. Conservation of mass, in cylindrical coordinates, is expressed by the equation:

$$\frac{\partial}{\partial r^*}(r^*u^*) + \frac{\partial}{\partial z^*}(r^*w^*) = 0 \tag{1}$$

where mean turbulent radial and axial velocities are used. This equation holds in the fluid region defined by $0 \le r^* \le \delta^*$ and $0 \le z^* \le l_0$. To satisfy equation (1), we write expressions for the axial and radial velocities:

$$w^*(r^*, z^*, t^*) = Q(t^*)f(r^*/\delta^*)/\delta^{*2}(z^*, t^*), \tag{2}$$

$$u^*(r^*, z^*, t^*) = (r^*/\delta^*)w^*(\partial\delta^*/\partial z^*). \tag{3}$$

Radial symmetry and the no-slip condition on the solid/liquid interface together imply that $f(R)$ is an even function of R and that $f(1) = 0$. Without loss of generality, we may impose a further condition on f, namely

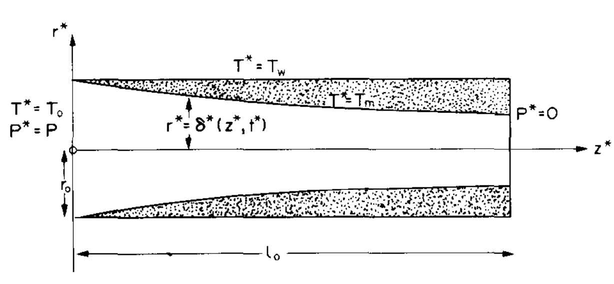

FIG. 1.

$$\frac{1}{\delta^{*2}}\int_0^{\delta^*} 2r^*f(r^*/\delta^*)\,\mathrm{d}r^* = 1, \quad (4)$$

or alternatively

$$\int_0^1 2Rf(R)\,\mathrm{d}R = 1. \quad (5)$$

This ensures that the bulk mean axial velocity, defined by

$$\bar{w}^*(z^*, t^*) = \frac{1}{\delta^{*2}}\int_0^{\delta^*} 2r^*w^*\,\mathrm{d}r^* \quad (6)$$

is given by the expression

$$\bar{w}^* = Q(t^*)/\delta^{*2}(z^*, t^*). \quad (7)$$

The local Reynolds number, based upon the local diameter (i.e. $2\delta^*$) is then expressed as

$$Re = 2\delta^*\bar{w}^*/\nu = 2Q(t^*)/\nu\delta^*. \quad (8)$$

Now most of the terms in the axial momentum equation, being either of the order of the Stefan number or inversely proportional to the square of the initial Peclet number, are neglected. There remains the standard formula relating the pressure gradient to the bulk axial velocity

$$\partial p^*/\partial z^* = -\,\rho f'\bar{w}^{*2}/4\delta^* \quad (9)$$

where f' is the friction factor. For laminar flow, the friction factor is inversely proportional to the Reynolds number (Hagen–Poiseuille law for Reynolds number less than about 2000), whereas for turbulent flow the situation is more complex. For a Reynolds number in the range from about 2000–10^5, a suitable law is the well known Blasius formula,

$$f' = 0.3164\,Re^{-1/4} \quad (10)$$

which will be used in equation (9). More accurate expressions for friction laws do exist, valid over larger ranges of Reynolds number, but these are usually implicit in nature, making analysis difficult. Numerical analysis can be performed with some confidence on pipe configurations where the initial Reynolds number lies within the range of validity of (10). Above this range results may be suspect, but no more so than the tacit assumption that the Blasius friction law holds good for pipes of variable radius. Below this range, the laminar analysis as presented in Part I [8] may be used. On using the above definitions of Reynolds number and friction factor in equation (9), we recover

$$\partial p^*/\partial z^* = -0.3164\,(2^{-9/4})\rho\nu^{1/4}Q^{7/4}(t^*)\delta^{*-19/4}. \quad (11)$$

The pressure must satisfy the boundary conditions $p^*(0, t^*) = P$ and $p^*(l_0, t^*) = 0$ appropriate to the assumption of a constant pressure drop maintained over the pipe length. There are thus two boundary conditions for a 1st-order differential equation and so the solution is:

$$p^*(z^*, t^*) = P\,\frac{\int_{z^*}^{l_0}\delta^{*-19/4}\,\mathrm{d}x}{\int_0^{l_0}\delta^{*-19/4}\,\mathrm{d}x} \quad (12)$$

where $Q(t^*)$ is also determined by this equation and is given by

$$Q(t^*) = (8\nu)^{3/7}\left[P/(0.3164\,\rho\nu\int_0^{l_0}\delta^{*-19/4}\,\mathrm{d}z^*)\right]^{4/7}. \quad (13)$$

In this way, we see that the velocity field is defined by the fixed initial pressure drop. It is clear that as the crust thickens (i.e. as δ^* decreases) the axial inlet velocity will decrease although the axial velocity as a whole will increase along the pipe. The decrease of the inlet velocity is due to the increasing viscous resistance whilst the narrowing of the effective radius along the pipe accounts for the increase in speed. This approach to inlet conditions is fundamental and is the main reason why it is possible to obtain blockage criteria. Other models have proposed that it is the inlet axial velocity which, as opposed to the pressure drop, is held constant, an untenable position we believe since the axial velocity must then increase indefinitely as the pipe blocks.

THERMODYNAMICS

For a description of the heat transport we use the convective diffusion equation, ignoring axial conduction and the time derivative since they are of the order of the inverse square of the initial Peclet number and the order of the Stefan number respectively. Thus

$$u^*\frac{\partial T^*}{\partial r^*} + w^*\frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*g^*\frac{\partial T^*}{\partial r^*}) \quad (14)$$

where mean turbulent quantities are again used and g^* is the radially dependent turbulent thermal diffusivity. This diffusivity is typically the same as the laminar diffusivity near boundaries but is higher near the centre of the flow, see e.g. Kays [9]. Unfortunately, from the points of view of simplicity, g^* also depends upon the Prandtl and Reynolds numbers.

In the solid region, defined by $\delta^* \le r^* \le r_0$ the velocities u^* and w^* are zero and g^* is constant, hence the temperature in the crust is given by:

$$T^* = T_w + (T_m - T_w)\frac{\ln(r_0/r^*)}{\ln(r_0/\delta^*)} \quad (15)$$

satisfying the isothermal boundary conditions on the pipe wall and the solid/liquid interface. The fluid region needs more detailed consideration and to this end it is convenient to non-dimensionalize the equations:

$$T^* = T_m + (T_0 - T_m)\theta(r, z, t), \quad (16)$$

$$\delta^* = r_0\delta(z,t), \quad (17)$$

$$r^* = r_0 r, \quad (18)$$

$$z^* = r_0 Pe_0 z, \quad (19)$$

$$t^* = r_0^2 t/2\kappa\,St, \quad (20)$$

$$g^* = \kappa g(r/\delta) \quad (21)$$

which gives

$$\frac{1}{2}\frac{1}{\delta^2}f(r/\delta)\left\{\frac{r}{\delta}\frac{\partial\delta}{\partial z}\frac{\partial\theta}{\partial r}+\frac{\partial\theta}{\partial z}\right\} = \frac{\kappa r_0 Pe_0}{2Q(t)}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(rg\frac{\partial\theta}{\partial r}\right) \quad (22)$$

where θ is subject to the boundary conditions

$$\theta(r,0,t)=1, \quad (23)$$

$$\theta(\delta,z,t)=0. \quad (24)$$

Certain conditions on $f(r/\delta)$ have already been noted and those on $g(r/\delta)$ are that $g(0)>1$ and $g(1)=1$ with $\mathrm{d}g(x)/\mathrm{d}x\le 0$. Note that Re_0 (the Reynolds number at $t=0$) is given by

$$Re_0 = 2Q(0)/\nu r_0 = \left[\frac{64}{0.1364}\frac{\kappa l_0}{\nu r_0}Gz\right]^{4/7}, \quad (25)$$

and so if we define the quantity

$$D(t)=Q(0)/Q(t) \quad (26)$$

we have that

$$D(t)=\left[\frac{1}{\alpha}\int_0^\alpha \delta^{-19/4}(z,t)\,\mathrm{d}z\right]^{4/7} \quad (27)$$

where

$$\alpha = l_0/r_0Pe_0 = \left[\frac{64}{0.3164}Gz\right]^{-4/7}\left[\frac{\kappa l_0}{\nu r_0}\right]^{3/7}. \quad (28)$$

There is a further interface condition to (14), viz.

$$\rho L\frac{\partial\delta^*}{\partial t^*} = \left(k_s\frac{\partial T^*}{\partial r^*}\right)_{r^*=\delta^{*+}} - \left(k_f\frac{\partial T^*}{\partial r^*}\right)_{r^*=\delta^{*-}} \quad (29)$$

which in dimensionless form now becomes, using (15),

$$2\frac{\partial\delta}{\partial t} = \frac{B}{\delta\ln\delta} - \left(\frac{\partial\theta}{\partial r}\right)_{r=\delta^-} \quad (30)$$

where the dimensionless freezing parameter is

$$B = k_s(T_m - T_w)/k_f(T_0 - T_m). \quad (31)$$

To remove the difficulty of having to numerically track a moving boundary, we transform the independent variables by writing $R = r/\delta$ (and keeping z as it is) to obtain the 'crust free' version of (22):

$$\frac{1}{2}f(R)\frac{\partial T}{\partial z} = D(t)\cdot\frac{1}{R}\frac{\partial}{\partial R}\left(Rg(R)\frac{\partial T}{\partial R}\right) \quad (32)$$

where $T(R,z)=\theta(R\delta,z,t)$ subject to

$$T(R,0)=1, \quad (33)$$

$$T(1,z)=0 \quad (34)$$

and (30) becomes

$$2\delta\frac{\partial\delta}{\partial t} = \frac{B}{\ln\delta} - \left(\frac{\partial T}{\partial R}\right)_{R=1}. \quad (35)$$

In practice, we look for a solution to (35) subject to (32)–(34) rather than vice versa. Further conditions on (35) are:

$$\delta(z,0)=\delta(0,t)=1. \quad (36)$$

THE TURBULENT GRAETZ PROBLEM

Equation (32) can be solved by writing:

$$T(R,z)=\sum_{n=0}^{\infty} C_nR_n(R)\exp\left[-\lambda_n^2 D(t)z\right], \quad (37)$$

where from (33)

$$\sum_{n=0}^{\infty} C_nR_n(R)=1 \quad (38)$$

and $R_n(R)$ satisfies the eigenvalue problem:

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[Rg(R)\frac{\mathrm{d}R_n}{\mathrm{d}R}\right]+\frac{1}{2}\lambda_n^2 Rf(R)R_n(R)=0 \quad (39)$$

with initial conditions

$$R_n(0)=1 \text{ and } R_n'(0)=0 \quad (40)$$

where the eigenvalues λ_n are such that they must satisfy (34), i.e. we must have that

$$R_n(1)=0. \quad (41)$$

Solutions to the system of equations represented by (39) have been generated numerically for the first few values of n (see [4–6]) and for larger eigenvalues, asymptotic expressions have been derived. We use the notation and results presented in [4] and [5]:

$$G\lambda_n = (n+2/3)+\left[c\ln(n+2/3)\pi + 7/18\pi\right]/2\pi(n+2/3) \quad (42)$$

where both G and c depend upon the Reynolds number of the flow and the Prandtl number of the fluid through the turbulent functions f and g in (39). Equation (35) requires a series for $\partial T/\partial R$ evaluated at $R=1$. This is obtained from (37) as the series expansion

$$\left(-\frac{\partial T}{\partial R}\right)_{R=1} = \sum_{n=0}^{\infty} 2A_n\exp\left[-\lambda_n^2 D(t)z\right] \quad (43)$$

where

$$A_n = -C_nR_n'(1)/2 \quad (44)$$

and the asymptotic (large n) expression for A_n is

$$A_n \sim \frac{(H/G^2)^{1/3}}{(G\lambda_n)^{1/3}}\times\frac{3^{5/6}}{2\pi}\times\frac{\Gamma(2/3)}{\Gamma(1/3)} \times\left\{1-\frac{1}{2\pi(G\lambda_n)^2}\left[c(\ln G\lambda_n\pi - 1)+\frac{7}{18\pi}\right]\right\} \quad (45)$$

where H is defined as

$$H = (0.3164/32)\,Re^{3/4} = -\frac{1}{2}\left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=1} = Ref'/32. \quad (46)$$

The quantities c and G which are present in these asymptotic formulae have been listed for certain discrete values of Re and Pr in [5]. It is worth noting that the expression for the interface heat flux will be calculated as a function of z and t and since the flux depends upon the flow, the coefficients and eigenvalues must be calculated as functions of the local Reynolds number. Thus there is no 'once and for all' calculation of the heat flux. We have already remarked that the axial velocity, and so by implication the local Reynolds number, is an increasing function of z but a globally decreasing function of t. In fact, numerical evidence indicates that the dependence of the Reynolds number on z is rather weak and so it might be possible to avoid the recalculation of the A_n and λ_n at every axial grid point by using a time dependent, axially averaged, Reynolds number.

NUMERICAL CALCULATIONS

Because of the four parameter nature of the system (i.e. Re_0, Pr, α and B) it would be difficult to present results which would be meaningful in a general sense. Instead, the method of solution will be outlined, with an example, and the reader should refer to an earlier paper [10] by the authors which describes a computer programme written in FORTRAN designed to calculate the time development of crusts with laminar or turbulent flow given the physical parameters of the system.

The first step is to calculate coefficients for a pair of 4th-order polynomial functions, dependent upon Re, for the turbulent flow parameters c and G. These are calculated from knowledge of the Prandtl number and using the data found in [4] and [5]. Subsequent values of these parameters may be calculated without referring again to the Prandtl number. At the start of the experiment, the temperature gradient is zero and so equation (35) can be integrated exactly, for small t, to give

$$2Bt = 1 - \delta^2 + \delta^2 \ln(\delta^2) \tag{47}$$

which can be inverted for small t:

$$\delta(z,t) \approx 1 - (Bt)^{1/2} \tag{48}$$

to give an initial evenly distributed 'thin-ice' crust, except at $z = 0$ where (36) always holds. The quantity $D(t)$ can then be calculated by Simpson's Rule. Following this, it is necessary to calculate Re at every axial grid point, from the relation:

$$Re = 2Q(t)/\nu r_0 \delta(z,t) = Re_0/\delta(z,t)D(t) \tag{49}$$

Table 2. Dimensionless parameters

	Run (1)	Run (2)	Run (3)	Run (4)
B	0.04297	0.08594	0.12891	0.17189
α	0.1289	0.1289	0.1289	0.1289
Pe_0	77.6	77.6	77.6	77.6
Pr	0.01049	0.01049	0.01049	0.01049
Gz	30.598	30.598	30.598	30.598
St	0.0262	0.0262	0.0262	0.0262
Re_0	7399	7399	7399	7399

and hence it is then possible to calculate c and G from which we obtain as many of the values of A_n and λ_n that are necessary for the calculation of the interface heat flux. A 4th-order Runge–Kutta process is then used to calculate new values for δ and the procedure continues with the next calculation for $D(t)$. This is done until either a steady state is reached or until the pipe blocks.

For the purposes of numerical analysis, steady state will be said to have been reached if the magnitude of the crust velocity at the pipe exit becomes smaller than some appropriate value (typically about 10^{-5} in dimensionless units). Blockage can be defined when the radial position of the crust goes negative (as it may in this model) but in most cases the calculations will stop since near blockage, more detailed analysis should be applied (see e.g. [11]). Instead, we note that the pipe will block and this is signalled either by $D(t)$ becoming too large or by the fact that the crust velocity at the exit, having reached a non-zero minimum absolute value, is beginning to increase again. In order to illustrate the behaviour of the crust, we will provide four specific examples to demonstrate behaviour with varying wall temperature. Table 1 lists the inputs for these examples while Table 2 lists the associated dimensionless parameters. Figures 2 and 3 show the growth of the crusts at the pipe exit with time and the rates of crust growth respectively. From these phase diagrams it is easy to see which will block well before the same information can be deduced by looking at the crust alone.

The predicted blockage time of run (4), from Fig. 2, is about 61 s, although it should be remembered that this

Table 1. Inputs

Unit	Run (1)	Run (2)	Run (3)	Run (4)
Pipe length (cm)	20	20	20	20
Pipe radius (cm)	2	2	2	2
Pressure drop ($kgm^{-1}s^{-2}$)	452.4	452.4	452.4	452.4
Density (kgm^{-3})	7232	7232	7232	7232
Kinematic viscosity ($\times 10^{-6} m^2 s^{-1}$)	4.63	4.63	4.63	4.63
Solid conductivity ($Wm^{-1}K^{-1}$)	198.7	198.7	198.7	198.7
Fluid conductivity ($Wm^{-1}K^{-1}$)	231.2	231.2	231.2	231.2
Specific heat ($Jkg^{-1}K^{-1}$)	72.4	72.4	72.4	72.4
Latent heat ($kJkg^{-1}$)	276	276	276	276
Fusion temperature (K)	1800	1800	1800	1800
Inlet temperature (K)	1900	1900	1900	1900
Wall temperature (K)	1795	1790	1785	1780
Initial mass flow rate (kgs^{-1}) [derived = $\pi\rho Q(0)$]	7.783	7.783	7.783	7.783

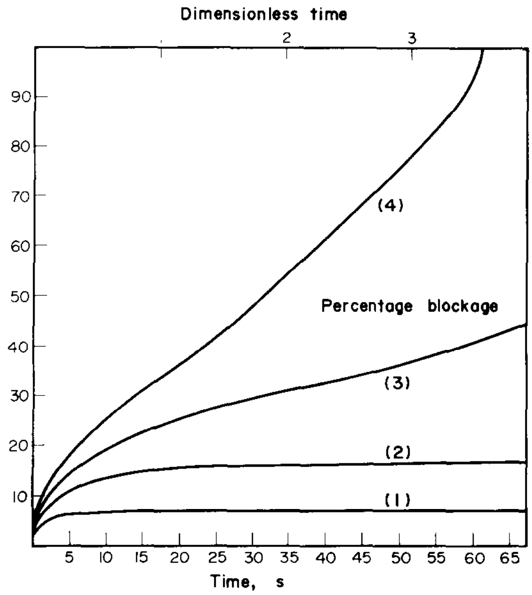

FIG. 2

is not to be taken too seriously since the model would no longer be accurate. It does, however, provide a reasonable minimum estimate. Run (2) seems to be near critical since it has not succeeded in satisfying the criterion for steady state in the time available although the exit crust has remained at 16.6% blocked for over 10 s with a value for $D(t) = 1.288$ for considerably longer. Run (1) has stabilised at 7% blockage after just under 35 s with $D = 1.108$. Another run, not shown in the figures since differentiation between them would be difficult, was performed with exactly the same inputs as run (1) but with double the pressure drop. In this case, a steady state blockage value was 7.2% with $D = 1.1006$ having taken just over 45 s to reach steady state.

CONCLUSIONS

This model should provide a reasonable description of the solidification of liquids. It is most suitable for those situations where unblocked steady states are reached where it can be used to describe the entire process. In the case of blockage, the entire process cannot be described with it since closure requires a more refined model as mentioned earlier. In addition, there is the problem, which does not arise in turbulent flow, that as the pipe progresses towards blockage the Reynolds number must eventually drop below the turbulent/laminar transition region. It is not clear whether in this case the flow remains turbulent but this has been considered as certain parameters (viz. c, G and H) are limited by their laminar values in the computation which should ensure that the description is accurate for as long as possible. However, we do not regard the details of blockage as particularly important but rather do we maintain that blockage will occur if there are no steady state solutions.

More accurate models would naturally include variable viscosities and densities and possibly more realistic fluids where no single melting point exists but rather a melting range over which the mechanism of latent heat would operate.

REFERENCES

1. J. Szekely and S. T. DiNovo, Thermal criteria for tundish nozzle or taphole blockage, *Met. Trans.* **5**, 747–754 (1974).
2. S. B. Thomason, J. C. Mulligan and J. Everhart, The effect of internal solidification on turbulent flow heat transfer and pressure drop in a horizontal tube, *Trans. A.S.M.E. J. Heat Transfer* **100**, 387–394 (1978).
3. J. R. Sellars, M. Tribus and J. S. Klein, Heat transfer to

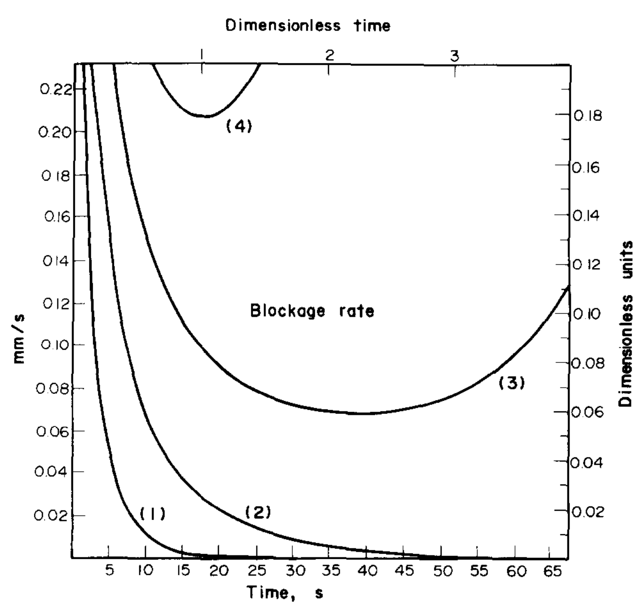

FIG. 3

laminar flow in a round tube or flat conduit—The Graetz problem extended, *Trans A.S.M.E.* **78C**, 441–481 (1956).
4. C. A. Sleicher, R. H. Notter and M. D. Crippen, A solution to the turbulent Graetz problem by matched asymptotic expansions—I. The case of uniform wall temperature, *Chem. Engng. Sci.* **25**, 845–857 (1970).
5. R. H. Notter and C. A. Sleicher, A solution to the turbulent Graetz problem III Fully developed and entry region heat transfer rates, *Chem. Engng. Sci.* **27**, 2073–2093 (1972).
6. A. A. Shibani and M. N. Özisik, Freezing of liquids in turbulent flow inside tubes, *Can. J. Chem. Engng* **55**, 672–677 (1977).
7. C. Cho and M. N. Ozisik, Transient freezing of liquids in turbulent flow inside tubes, *Trans. A.S.M.E. J. Heat Transfer* **101**, 465–468 (1979).
8. P. Sampson and R. D. Gibson, A mathematical model of nozzle blockage by freezing, *Int. J. Heat Mass Transfer* **24**, 231–241 (1981).
9. W. M. Kays, Heat transfer: turbulent flow inside smooth tubes, in *Convective Heat and Mass Transfer*, Ch. 9. McGraw-Hill TMH Edition (1975).
10. P. Sampson and R. D. Gibson, Solidification of a liquid metal flowing through a circular pipe—A prediction of nozzle blockage, *Adv. Engng Software* **3**, 17–25 (1981).
11. D. S. Riley, F. T. Smith and G. Poots, The inward solidification of spheres and circular cylinders, *Int. J. Heat Mass Transfer* **17**, 1507–1516 (1974).

APPENDIX

We examine the existence of stable steady state solutions to eqn. (35) by setting the right-hand side equal to zero. It will be shown that there are no, one or two solutions which satisfy (35) and (27) dependent upon the values found for α, B, Re_0 and Pr. The analysis is similar to that described in [8] but since exponents of 4/7 appear in the turbulent model it is not immediately clear that singularities will not arise.

If we assume a steady state solution for δ, denoted by $\delta_s(z)$ and a corresponding steady state value of D_s for $D(t)$ then from (35) and (43) we can write

$$\delta_s(z) = \exp\left[-B/\sum_{n=0}^{\infty} 2A_n \exp(-\lambda_n^2 D_s z)\right] \tag{A.1}$$

and we can define a function μ:

$$\mu(D_s) = D_s^{7/4} - \frac{1}{\alpha}\int_0^{\alpha} \delta_s^{-19/4}(z)\,dz. \tag{A.2}$$

By substituting (A.1) into (A.2) we can plot, for fixed α, Pr and Re_0, a graph of μ against D_s for parameter B. An accurate plot can be obtained only by including the dependence of A_n and λ_n upon D_s but the general shape of the curves will be unaffected. To obtain $\mu(D_s)$ accurately, iteration would be performed with D_s and $\delta_s(z)$ and it can be shown that,

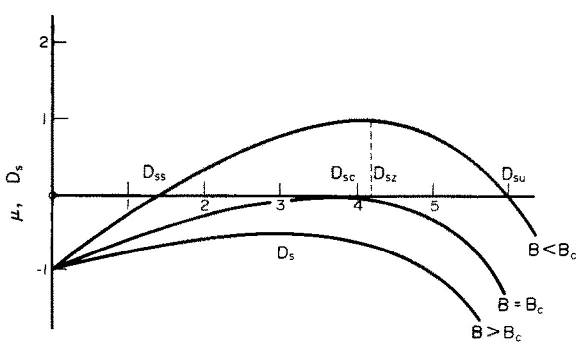

FIG. A1

provided that the initial estimate of D_s is sufficiently close to unity, the iterative scheme would converge, if convergent, to a steady state solution.

Figure A1 shows a sketch of $\mu(D_s)$ for variable B and fixed α, Pr and Re_0 showing the three different cases that might arise. In particular we will examine the case appropriate to two steady state solutions in the following analysis.

On referring to the figure, we wish to show that D_{ss} is a stable solution and that D_{su} is unstable. First we shall write

$$\frac{\partial \delta^2}{\partial t} = \frac{2B}{\ln \delta} + \hat{f}(z, t) \tag{A.3}$$

for $0 \le z \le \alpha$ and for $t \ge$ where the function $\hat{f}$ behaves thus:

$$\hat{f} = O(z^{-1/3}) \text{ as } z \to 0, \tag{A.4a}$$

$$\hat{f} = O[\exp(-kz)] \text{ as } z \to \infty, \tag{A.4b}$$

$$\partial \hat{f}/\partial z < 0 \text{ for all } z > 0 \tag{A.4c}$$

and δ^2 is O(1) as $z \to 0$. Equation (A.1) is then

$$\delta_s^2(z) = \exp[-2B/\hat{f}_s(z)] \tag{A.5}$$

where $\hat{f}_s(z)$ is the assumed steady state value of $\hat{f}(z, t)$. We introduce linear perturbations about the steady state by writing

$$D(t) = D_s + d(t); \delta^2(z, t) = \delta_s^2(z) + \hat{g}(z)\exp(\sigma t) \tag{A.6}$$

where σ is a growth rate away from steady state. Substituting into (A.3) and (27) gives, after linearization,

$$(\sigma + 2B/\delta_s^2 \ln^2 \delta_s^2)\hat{g}(z) = z e^{-\sigma t} d(t)(d\hat{f}_s/dz)/D_s, \tag{A.7}$$

$$d(t) = \frac{-19}{14} e^{\sigma t} D_s^{-3/4} \frac{1}{\alpha}\int_0^{\alpha} \hat{g}(z)\delta_s^{-27/4}(z)\,dz \tag{A.8}$$

where $d\hat{f}/dz$ in (A.7) is an approximation since the small axial variations in the A_n and λ_n have been neglected as irrelevant to the analysis. If we let

$$d_1 = \frac{1}{\alpha}\int_0^{\alpha} \hat{g}(z)\delta_s^{-27/4}(z)\,dz \tag{A.9}$$

then (A.7) and (A.8) give

$$(\sigma + 2B/\delta_s^2 \ln^2 \delta_s^2)\hat{g}(z) = \frac{-19}{14} D_s^{-7/4} d_1 z(d\hat{f}_s/dz) \tag{A.10}$$

and so we obtain

$$\hat{g}(z)\delta_s^{-27/4} = -\frac{19}{14} D_s^{-7/4}\, z(d\hat{f}_s/dz)/(\sigma + 2B/\delta_s^2 \ln^2 \delta_s^2) \tag{A.11}$$

which can then be integrated, excluding the trivial solution where $d_1 = 0$, to give the condition

$$1 = -\frac{19}{14} D_s^{-7/4} \frac{1}{\alpha}\int_0^{\alpha} \frac{z(d\hat{f}_s/dz)\,dz}{\delta_s^{27/4}(\sigma + 2B/\delta_s^2 \ln^2 \delta_s^2)} \tag{A.12}$$

from which σ can be, in principle, obtained. Now, on substituting (A.5) into (A.12) we obtain

$$1 = -\frac{19}{14} D_s^{-7/4} \frac{1}{\alpha}\int_0^{\alpha} \frac{2Bz(d\hat{f}_s/dz)\exp(27B/4\hat{f}_s)\,dz}{[2\sigma B + \hat{f}_s^2 \exp(2B/\hat{f}_s)]} \tag{A.13}$$

Clearly, if $\sigma = 0$ then this reduces to

$$1 = -\frac{19}{14} D_s^{-7/4} \frac{2B}{\alpha}\int_0^{\alpha} \exp(19B/4\hat{f}_s) z\, d\hat{f}_s/\hat{f}_s^2 \tag{A.14}$$

which has the solution

$$D_s = \left(\frac{4}{11}\right)^{4/7} \delta_s^{-19/4}(\alpha) = \left(\frac{4}{11}\right)^{4/7} \exp[19B/7\hat{f}_s(\alpha)] \tag{A.15}$$

which is also obtainable directly from (A.2) and so (A.15) satisfies the equation $\mu = d\mu/dD_s = 0$ which means that the case $\sigma = 0$ represents the critical case given by D_{sc} in Fig. A1.

When eqn. (A.2) has two distinct roots, D_{ss} and D_{su}, it can be seen from the figure that there exists a value D_{sc} such that $D_{ss} < D_{sz} < D_{su}$ which will satisfy the condition $d\mu/dD_s = 0$ but will not satisfy $\mu = 0$. We know that since $d\mu/dD_s > 0$ at $D_s = D_{ss}$ then

$$D_{ss} > \left(\frac{4}{11}\right)^{4/7} \exp[19B/7\hat{f}_{ss}(\alpha)] \qquad (A.16)$$

and also

$$D_{su} < \left(\frac{4}{11}\right)^{4/7} \exp[19B/7\hat{f}_{su}(\alpha)] \qquad (A.17)$$

where $\hat{f}_{ss}(\alpha)$ and $\hat{f}_{su}(\mu)$ are the corresponding appropriate values of $\hat{f}_s(\alpha)$. If we assume that σ is positive then (A.13) gives

$$1 = \frac{-19B}{7\alpha D_{ss}^{7/4}} \int_0^{\alpha} \frac{z(d\hat{f}_{ss}/dz)\exp(27B/4\hat{f}_{ss})\,dz}{2B\sigma + \hat{f}_{ss}^2 \exp(2B/\hat{f}_{ss})}$$

$$< \frac{-19B}{7\alpha D_{ss}^{7/4}} \int_0^{\alpha} z(d\hat{f}_{ss}/dz)\exp(19B/4\hat{f}_{ss})\,dz/\hat{f}_{ss}^2$$

$$= \frac{4}{7} D_{ss}^{-7/4} \exp[19B/4\hat{f}_{ss}(\alpha)] - \frac{4}{7} \qquad (A.18)$$

since $d\hat{f}_{ss}/dz < 0$, hence

$$D_{ss} < \left(\frac{4}{11}\right)^{4/7} \exp[19B/7\hat{f}_{ss}(\alpha)] \qquad (A.19)$$

which contradicts (A.16) and so σ must be negative if D_{ss} is to satisfy (A.13) which means that D_{ss} represents a stable solution. If σ is negative then the condition for D_{su} to satisfy (A.13) becomes, by exactly the same argument,

$$1 > \frac{4}{7} D_{su}^{-7/4} \exp[19B/4\hat{f}_{su}(\alpha)] - \frac{4}{7} \qquad (A.20)$$

where the inequality has been reversed since

$$2B\sigma + \hat{f}_{su}^2 \exp(2B/\hat{f}_{su})$$

is now less than $\hat{f}_{su}^2 \exp(2B/\hat{f}_{su})$ due to the sign of σ. Equation (A.20) contradicts (A.17) and so D_{su} is an unstable solution.

UN MODELE MATHEMATIQUE DU BLOCAGE DE TUYERE PAR LE GEL II.—ECOULEMENT TURBULENT

Résumé—Un métal fondu s'écoulant dans un tuyau cylindrique est solidifié par la paroi froide du tuyau et une croûte solide est formée sur la paroi. On présente un modèle pour décrire la croissance de cette croûte, l'écoulement étant turbulent. Des critères qui déterminent s'il y a blocage ou non sont étudiés et quelques résultats de l'analyse théorique sont présentés pour des écoulements avec un nombre de Reynolds compris entre 2000 et 100.000 et un nombre de Prandtl entre 0,007 et 0,1.

EIN MATHEMATISCHES MODELL FÜR DIE BLOCKIERUNG EINER MÜNDUNG DURCH GEFRIEREN

Zusammenfassung—Heißes, geschmolzenes Metall, das durch ein zylindrisches Rohr fließt, erstarrt an den kalten Rohrwänden und bildet dort eine feste Kruste. Ein Modell wird angegeben, das für turbulente Rohrströmung die Wachstumsrate dieser Kruste beschreibt. Die Kriterien wurden untersucht, die angeben, ob das Rohr zufriert oder nicht. Einige Ergebnisse der theoretischen Berechnung werden angegeben. Sie gelten für Strömungen mit Reynolds-Zahlen zwischen 2000 und 100 000, die zu Flüssigkeiten mit Prandtl-Zahlen im Bereich zwischen 0,007 bis 0,1 passen.

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ЗАПИРАНИЯ СОПЛА ПРИ ОСТЫВАНИИ РАСПЛАВЛЕННОГО МЕТАЛЛА. ЧАСТЬ II – ТУРБУЛЕНТНОЕ ТЕЧЕНИЕ

Аннотация – При течении в цилиндрической трубе горячий расплавленный металл застывает у холодных стенок трубы, в результате чего на стенках образуется твердая корка. Для описания процесса образования такой корки представлена модель, в которой предполагается, что течение в трубе является турбулентным. Приведены критерии для определения возможности возникновения запирания, и представлены некоторые результаты теоретического анализа для течений со значениями числа Рейнольдса, лежащими в диапазоне 2000–100 000, что соответствует жидкостям со значениями числа Прандтля от 0,007 до 0,1.